Frir M-4

Differential Equation

A first order first degree differential eqn be expressed as $Mdx + Ndy = 0$ (or Const)
M & N be either fn of x or fn of y, fn of xy or Constant.
It being called exact differential eqn iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If not then called non exact differential eqn. A non exact differential eqn can be transferred into exact differential eqn by multipling a factor called I.F (Integrating factor). [Non-exact diffal eqn $\xrightarrow{\times(\text{I.F})}$ exact differential eqn]

Linear differential eqn

A diffal eqn as a form of $\frac{dy}{dx} + Py = Q$
P & Q be fn of x or Constant called Linear diffal eqn

w. r. to x

$$\frac{dy}{dx} e^{\int P dx} + P y e^{\int P dx} = Q e^{\int P dx}.$$

$$\frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

Int. b. side w.r.t. x.

$$y e^{\int P dx} = \int Q e^{\int P dx} + c.$$

Bernoullie's eqn : A differential eqn be formed as $\frac{dy}{dx} + Py = Qy^n$.

$$y^{-n}\frac{dy}{dx} + P\boxed{y^{1-n}} = Q. \quad \text{---} (1) \quad n \in \mathbb{Z}$$

$$y^{1-n} = z.$$

$$(1-n) y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}.$$

From (1)

$$\frac{1}{(1-n)}\frac{dz}{dx} + P.z = Q$$

$$\frac{dx}{dy} + Px = Q$$ — P, Q fn. of y or const.

$$\frac{dy}{dx} + Py = Q$$ → P, Q fn. of x or const.

It is a non exact diff^{al} eq^{n}.

$$dy + (Py - Q)\,dx = 0$$

$$M\,dy + N\,dx$$

$M = 1$ | $N = Py - Q$,

$\frac{\partial M}{\partial y} = 0$. | $\frac{\partial N}{\partial x}$ = fn. of x or Constant $\neq 0$ in all time.

i.e. non exact diff^{al} eq^{n}.

→ Exact $I.F = e^{\int P\,dx}$.

P fn. of x or const. Treat as definite integral. So $\nexists$ any integral Constant.

$$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

It is Linear eqn.

$$\therefore \text{I.F.} = e^{\int P(1-n)\,dz}$$

$$\frac{dz}{dx}\,e^{\int (1-n)\,dz} + P(1-n)\,z\,e^{\int P(1-n)\,dz} + Q(1-n)\,e^{\int P(1-n)\,dz}$$

$$\frac{d}{dx}\left\{ z\, e^{\int P(1-n)\,dz} \right\} = Q(1-n)\,e^{\int P(1-n)\,dz}$$

Int.

$$e^{\int P(1-n)\,dz} \cdot z = \int Q(1-n)\,e^{\int P(1-n)\,dz}\,dz + C$$

* $\frac{dy}{dx} + xy = e^{-x^2/2}$ —— ①

$\frac{dy}{dx} + Py = Q$.

It is a Linear eqn. Comparing with given form

$P = x, \; Q = e^{-x^2/2}$

$\therefore$ I.F. $= e^{\int x\,dx} = e^{x^2/2}$.

'X' by I.F. in both side of ①.

$\frac{d}{dx}\left\{ y \cdot e^{x^2/2} \right\} = e^{-x^2/2} \cdot e^{x^2/2}$

$= 1$

Int. w.r.t. x

$y e^{x^2/2} = \int dx = x + c$

$\frac{dx^2}{dx} = 2x$

$2\int x\,dx$ $\frac{x^2}{2}$

$$\frac{dy}{dx} + y \cdot \cot x = \operatorname{cosec} x \quad \text{(1)}$$

is Linear eqn

Comparing with $\frac{dy}{dx} + Py = Q$

$P = \cot x, \quad Q = \operatorname{cosec} x.$

$I \cdot F = e^{\int \cot x \, dx}$

$\int \frac{\cos x}{\sin x} dx$

$= e^{\log|\sin x|}$

$\int \frac{D(\sin x)}{\sin x} dx$

$= \sin x$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)|$

It being multiplying with both sides of (A).

$$\frac{d}{dx}\{y \cdot \sin x\} = \operatorname{cosec} x \sin x$$

$= 1.$

Int. $y \sin x = x + c.$

$y = (x+c) \operatorname{cosec} x$

I.F.

$$z e^{-\frac{5}{2}x^2} = -5\int x^2 e^{-\frac{5}{2}x^2}\,dx + c$$

Now

$$\frac{1}{y^5} e^{-\frac{5}{2}x^2} = -5\int x^2 e^{-\frac{5}{2}x^2}\,dx$$

etc

SQ 7 × 11 ×

LQ 25 × 29 × 12 × 23 ×

8 ×

* $\frac{dy}{dx} + xy = x^2 y^{-6}$ —— (A)

$y^{-6} \frac{dy}{dx} + xy^{1-6} = x^2$

$y^{-6} \frac{dy}{dx} + xy^{-5} = x^2$ —— (A')

$y^{-5} = z$

$-5 y^{-6} \frac{dy}{dz} = \frac{dz}{dx}$.

$y^{-6} \frac{dy}{dz} = -\frac{1}{5} \frac{dz}{dx}$.

From (A')

$-\frac{1}{5} \frac{dz}{dx} + xz = x^2$.

$\frac{dz}{dx} - 5xz = -5x^2$. $\left[\frac{dy}{dx} + Py = Q\right]$

I.F. $= e^{-\int 5x\, dx} = e^{-\frac{5}{2}x^2}$

$\frac{d}{dx}\left(z \cdot e^{-\frac{5}{2}x^2}\right) = -5x^2 \cdot e^{-\frac{5}{2}x^2}$

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Linear eqn $\frac{dy}{dx} + Py = Q$.
P, Q fn. of independent variable ~~and~~ or Constant. It is a non exact diffl eqn [A first order first degree differential expressed as $Mdx + Ndy = 0$ is called exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. M & N be fn. of x or fn. of y or fn. of xy or Constant).

A nonexact $\xrightarrow{\times (I.F)}$ exact

For linear eqn $\frac{dy}{dx} + Py = Q$ (non exact)

$$I.F = e^{\int P dx}$$

$$e^{\int P dx}\frac{dy}{dx} + e^{\int P dx} Py = e^{\int P dx} \cdot Q$$

$$\frac{d}{dx}\left(y e^{\int P dx}\right) = e^{\int P dx} \cdot Q$$

Int. w.r.t. x.

$$y e^{\int P dx} = \int e^{\int P dx} \cdot Q + C.$$

C = I.C

$\frac{dx}{dy} + Px = Q \longrightarrow$ P, Q fn. of y or Constant

Rules of I.F :-

① If $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an I.F of that differential eqn.

② $Mdx + Ndy = 0$ is a first order first degree diff eqn. taken

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \text{ (only function of } x)$$

$$I.F = e^{\int f(x)\,dx}$$

③ $$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \phi(y) \text{ (fn. of } y \text{ only)}$$

$$I.F = e^{\int \phi(y)\,dy}$$

④ If $Mx - Ny \neq 0$, $\frac{1}{Mx - Ny}$ is an I.Factor.

⑤ $x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1} y^{\beta_1}(m_1 y\,dx + n_1 x\,dy) = 0$.

$x^{h} y^{k}$ is an I.F,

where $\frac{\alpha_1 + 1 + \alpha}{m} = \frac{k_1 + 1 + \beta}{n}$

$x\,dx = \frac{x^2}{2}$

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$$\frac{\lambda x\,[-(\alpha)]}{x_1} = \frac{K m+1+\beta}{m_1}$$

7 (i-J) $x^2y\,dx - (x^3+y^3)\,dy = 0$.

$M = x^2y, \quad N = -x^3+y^3$

$Mx = x^3y \qquad Ny = -x^3y - y^4$.

$Mx + Ny = -y^4 \neq 0$. is an I.F

$$-\frac{x^2y}{y^4}\,dx + \frac{x^3}{y^4}\,dy + \frac{y^3}{y^4}\,dy = 0 \qquad d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$$

$$+\frac{x^2}{y^4}\left\{-y\,dx + x\,dy\right\} + \frac{1}{y}\,dy = 0.$$

$$-\frac{x^2}{y^2}\left(\frac{-x\,dy + y\,dx}{y^2}\right) + \frac{1}{y}\,dy = 0.$$

$$-\left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) + \frac{1}{y}\,dy = 0$$

Int. $-\frac{x^3}{3y^3} + \log y = \log c$

$x^2y\,dx - (x^3+y^3)dy = 0$ Page No : Date :

$y(x^2dx - y^2dy) - y(x^3dy + 0\cdot dx)$
$= 0.$

$x^\alpha y^\beta(my\,dx + nx\,dy) + x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy) = 0$

Comparing $\alpha = 0$, $\beta = 1$, $n = -y$.

etc

R-II $\underbrace{(x^3+xy^4)}_{M}dx + \underbrace{2y^3}_{N}dy = 0.$

$M = x^3 + xy^4$ | $N = 2y^3$

$\frac{\partial M}{\partial y} = 4y^3x$ | $\frac{\partial N}{\partial x} = 0.$

$\frac{1}{N}(4y^3x - 0) = \frac{1}{2y^3}(4y^3x) = 2x$

$e^{\int 2x\,dx} = e^{x^2}$

$$z\int e^z dz - \int\left\{\frac{d}{dz}(z)\int e^z dz\right\}dz$$
$$= z e^z - \int e^z dz = e^z(z-1)$$

$$e^{x^2} x^3 dx + e^{x^2}\cdot x y^4 dx + 2y^3 e^{x^2} dy = 0.$$

$$\frac{1}{2} x^2 \cdot 2x \cdot e^{x^2} dx + \frac{1}{2} e^{x^2}\cdot 2x\, y^4 dx + e^{x^2}\cdot 2y^3 dy = 0.$$

$x^2 = z$, $2x\,dx = dz$

$$\cancel{\frac{1}{2}(z e^z)\, dz}$$

$$\frac{1}{2}\int z e^z dz + \frac{1}{2} y^4 \int e^z dz + \cancel{e^{x^2} 2y^3 dy}$$

$$\cancel{\frac{1}{2}\left(e^{z}\ldots\right)}$$

$$\frac{1}{2} e^z (z-1) + \frac{1}{2} y^4 \cdot e^z + e^{x^2}\cdot\frac{y^4}{4} = c_1$$

Put $x^2 = z$.

$$\cancel{Q-2}\quad (2xy^4 e^y + 2xy^3 + y)dx + (x^2y^4 e^y - x^2y^2 - 3x)dy = 0$$

$M = 2xy^4 e^y + 2xy^3 + y$, $\frac{\partial M}{\partial y} = 8xy^3 e^y + 6xy^2 + 1 + 2xy^4 e^y$.

$N = x^2 y^4 e^y - x^2 y^2 - 3x$, $\frac{\partial N}{\partial x} = 2xy^4 e^y - 2xy^2 - 3$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3 e^y + 8xy^2 + 4$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{y}. \quad I\cdot F = e^{\int \frac{4}{y} dy} = y^4$$

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... by I.F. then Int.

(4) $(x^2y^2+xy+1)y\,dx-(x^2y^2-xy+1)x\,dy$

$M=x^2y^3+xy^2+y$

$N=-x^3y^2+x^2y+x$.

$M_x=x^3y^3+x^2y^2+xy$

$N_y=-x^3y^3+x^2y^2+xy$

$2x^3y^3$

$\frac{x^2y^2}{2x^3y^3}dx+\frac{xy^2}{2x^3y^3}dx+\frac{y}{2x^3y^3}dx$

$=\frac{x^3y^2}{2x^3y^3}dy-\frac{x^2y}{2x^3y^3}dy+\frac{x}{2x^3y^3}dy$

Clairaut's form of diff'al eqn

~~$y = f(x) +$~~

$y = px + f(p)$ $p = \frac{dy}{dx}$

$y = mx + c$

It gives curve ~~gotted~~ from the tangent

Diff w.r.t. x in both side! –

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$\because \frac{dy}{dx} = p$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0.$$

either $\frac{dp}{dx} = 0 \Rightarrow p = c$

c – discriminant form $y = cx + f(c)$ → S.S

or $x + f'(p) = 0$

$x = -f'(p)$. Putting x in given diff'al eqn. Now relations is → G.S

$$y = px + \sqrt{a^2p^2+b^2}$$

diff w.r.t. x.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2}(a^2p^2+b^2)^{-1/2}$$

$\therefore \frac{dy}{dx} = p$

$$\frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2+b^2}}\right) = 0$$

$\frac{dp}{dx} = 0 \Rightarrow p = c$

C: dis $y = cx + \sqrt{a^2c^2+b^2}$

$$x + \frac{a^2p}{\sqrt{a^2p^2+b^2}} = 0$$

$$x = -\frac{a^2p}{\sqrt{a^2p^2+b^2}}$$

$$\frac{x}{a} = -\frac{a^2p}{\sqrt{a^2p^2+b^2}}$$

A

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Now

$$y = -p\frac{a^2p}{\sqrt{a^2p^2+b^2}} + \sqrt{a^2p^2+b^2}$$

$$= \cancel{\frac{-a^2p^2 + a^2p^2+b^2}{\sqrt{a^2p^2+b^2}}}$$

$$= \frac{b^2}{\sqrt{a^2p^2+b^2}}$$

$$\frac{y}{b} = \frac{b}{\sqrt{a^2p^2+b^2}} \quad \leftarrow \text{(B)}$$

$(A)^2 + (B)^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2p^2}{a^2p^2+b^2} + \frac{b^2}{a^2p^2+b^2}$$

$$= \frac{a^2p^2+b^2}{a^2p^2+b^2}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

* $y = px + \sin^{-1}p$. , $y =$