Sphere

1) The equation of a sphere having centre $(0,0,0)$ and radius r is $\underline{x^2+y^2+z^2=r^2}$

2) The equation of a sphere having centre (α,β,γ) & radius r is $\underline{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2}$

3) The general equation of a sphere is
$x^2+y^2+z^2+2gx+2fy+2hz+c=0$ whose centre is $(-g,-f,-h)$ & radius $=\sqrt{g^2+f^2+h^2-c}$ units.

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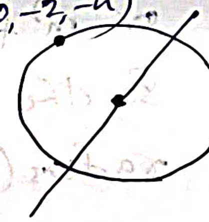
1) Show that the equation of the sphere passing through the pts $(0,-2,-4)$, $(2,-1,-1)$ & having its centre on the line $2x-3y=0=5y+2z$ is
$x^2+y^2+z^2-6x-4y+10z+12=0$

(0,-2,-4)



(2,-1,-1)

Sol The eqn. of the line is
$2x-3y=0=5y+2z$

$\therefore 2x=3y$ & $5y+2z=0$

or $\frac{x}{3}=\frac{y}{2}$ & $5y=-2z$ or $\frac{y}{2}=\frac{z}{-5}$

So, the eqn of the given st. line can be written as $\frac{x}{3}=\frac{y}{2}=\frac{z}{-5}=r$ (say) —(1)

Let $(3r, 2r, -5r)$ be the centre of the sphere.
Since the sphere passes through the pts $(0,-3-4)$ & $(2,-1,-1)$ so, we have,

$$\sqrt{(3r-0)^2+(2r+2)^2+(-5r+4)^2}=\sqrt{(3r-2)^2+(2r+1)^2+(-5r+1)^2}$$

or $8r+4+16-40r=-12r+4+4r+1-10r+1$

or $-32r+22r-4r=2-16$ or $-14r=-14$ or $\underline{r=1}$

So, the centre of the sphere is $(3,2-5)$

and radius of the sphere $= \sqrt{(3-0)^2+(2+2)^2+(-5+4)^2}$

$= \sqrt{9+16+1} = \sqrt{26}$

So, the eqn of the required sphere with centre $(3,2,-5)$ having radius $\sqrt{26}$ is

$$(x-3)^2+(y-2)^2+(z+5)^2 = (\sqrt{26})^2$$

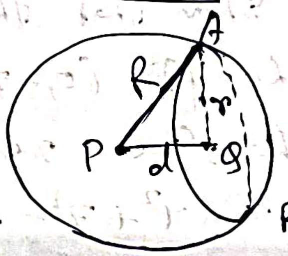
or $x^2-6x+9+y^2-4y+4+z^2+10z+25 = 26$

or $\underline{x^2+y^2+z^2-6x-4y+10z+12=0}$.

** 2) Find the centre & the radius of the circle $x^2+y^2+z^2-2y-4z-11=0$, $\underline{x+2y+2z=15}$

Sol The eqn of the sphere is

$x^2+y^2+z^2-2y-4z-11=0$ —(1)



or $x^2+(y-1)^2+(z-2)^2 = 11+1+4 = 16$

$R^2 = r^2+d^2$

So, the centre of the sphere (P) is $(0,1,2)$ & its radius $(R) = 4$ $(= AP)$

Now, the length of the perp^n from the centre $(0,1,2)$ of the sphere to the plane $x+2y+2z=15$

is, $d = \left|\dfrac{0+2+4-15}{\sqrt{1+4+4}}\right| = \left|\dfrac{-9}{3}\right| = 3$ $(= PQ)$

Now, we have; $AQ^2 = AP^2 - PQ^2 = 4^2-3^2 = 7$

So, the radius of the circle $(r) = \sqrt{7}$.

Now, the eqn of the line (PQ) perp^n to the plane $x+2y+2z=15$ —(2) & passes through the centre $(0,1,2)$ of the sphere is $\dfrac{x-0}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2} = r$ (say)

Let $Q = (r, 2r+1, 2r+2)$ be any pt on line PQ.

Since Q lies on plane (2), so we have,

$r+2(1+2r)+2(2r+2) = 15$

or $r+2+4r+4r+4=15$ or $9r=9$ or $\underline{r=1}$

So, the centre (Q) of the given circle is $(1,3,4)$

3) Find the value of h for which the plane $x+y+z=h$ is the tangent plane to the sphere $\underline{x^2+y^2+z^2=48}$

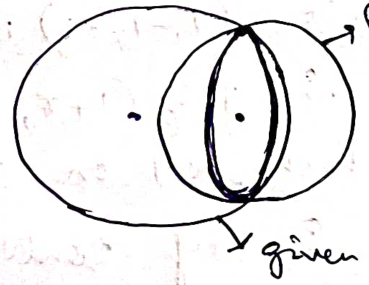
Sol The centre & radius of the sphere $\underline{x^2+y^2+z^2=48}$ —(1) are $(0,0,0)$ & $\sqrt{48}$ respectively.

Since the plane $\underline{x+y+z=h}$ —(2) is the tangent plane to the sphere (1). So,

Radius of the sphere = the perp^n dist of the centre of the sphere(1) from the plane (2).

So, we have, $\sqrt{48} = \pm\frac{h}{\sqrt{3}}$ or $\underline{h=\pm 12}$

** 4) Find the equation of the sphere having the circle $x^2+y^2+z^2-2x+4y-6z+7=0$, $2x-y+2z-5=0$ is a great circle.



Sol Let the eqn of the sphere containing the given circle be

$x^2+y^2+z^2-2x+4y-6z+7$
$+\lambda(2x-y+2z-5)=0$ —(1) $\lambda\neq 0$, λ = const

or $x^2+y^2+z^2+2(\lambda-1)x+2(2-\frac{\lambda}{2})y+2(\lambda-3)z+7-5\lambda=0$,

whose centre $=\{1-\lambda, \frac{\lambda}{2}-2, 3-\lambda\}$

Since the centre of the sphere coincide with the centre of the circle in case of a great circle,

So, the centre of the sphere $\{1-\lambda, \frac{\lambda}{2}-2, 3-\lambda\}$ must lie on the plane $\underline{2x-y+2z=5}$

So, $2(1-\lambda)-(\frac{\lambda}{2}-2)+2(3-\lambda)=5$

or $2-2\lambda-\frac{\lambda}{2}+2+6-2\lambda=5$ or $4\lambda+\frac{\lambda}{2}=5$

or $\frac{9\lambda}{2}=5$ or $\lambda=\frac{10}{9}$.

So the req. eqn. of the sphere is

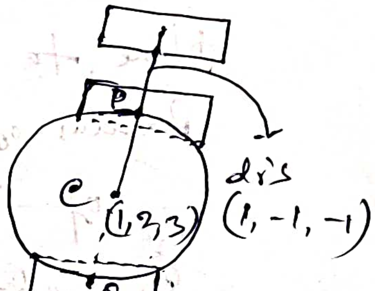
$$x^2+y^2+z^2-2x+4y-6z+7+\frac{10}{9}(2x-y+2z-5)=0$$

or $9(x^2+y^2+z^2)-18x+36y-54z+63+20x-10y+20z-50=0$

or $\underline{9(x^2+y^2+z^2)+2x+26y-34z+13=0}$ ✓

** 5) Find the eqn of the tangent planes to the sphere $x^2+y^2+z^2-2x-4y-6z+2=0$ parallel to the plane $x-y-z=0$

Sol The eqn of the sphere is



$$x^2+y^2+z^2-2x-4y-6z+2=0 \quad -(1)$$

or $x^2+y^2+z^2+2(-1)x+2(-2)y+2(-3)z+2=0$

So, centre of the sphere (1) = c(1,2,3) &

radius $=\sqrt{1+4+9-2}=\sqrt{12}$

Now, the given eqn of the plane is $x-y-z=0$ —(2)

So, the dr's of the normal to the plane (2) are $(1,-1,-1)$

So, the eqn of the line passes through $(1,2,3)$ having dr's $(1,-1,-1)$ is $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{-1}=r$ (say) $r\neq 0$

Let $P=(1+r,\ 2-r,\ 3-r)$ be any pt on the line, which lies on sphere (1)

So, by condition, $\sqrt{r^2+r^2+r^2}=\sqrt{12}$ or $3r^2=12$ $\therefore \underline{r=\pm 2}$

So, the sphere passes through two pts

$(1+2,\ 2-2,\ 3-2)$ & $(1-2,\ 2+2,\ 3+2)$ i.e $(3,0,1)$ & $(-1,4,5)$

So the eqn of the tangent planes passes through $(3,0,1)$ & $(-1,4,5)$ having dr's $(1,-1,-1)$ normal to the plane (2) are $1(x-3)-1(y-0)-1(z-1)=0$

& $1(x+1)-1(y-4)-1(z-5)=0$

i.e $\underline{x-y-z=2}$ & $\underline{x-y-z+10=0}$